Oscillator noise: a rigorous analysis including orbital fluctuations

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***Abstract*—We present results on the effect of orbital noise on the fluctuations spectrum of a noisy oscillator, exploiting a rigorous nonlinear perturbative analysis based on Floquet theory. We show that orbital fluctuations may significantly impact on the total oscillator noise spectrum.**

***Index Terms*—Circuit simulation, Autononmous systems, Nonlinear oscillators, Oscillator noise**

I. INTRODUCTION

Noise in free running oscillators has been studied for decades because of the significant impact on the performance of communication receivers. Therefore, the design of low noise autonomous circuits is a fundamental step in the optimization of high sensitivity telecommunication systems [1]. Oscillator noise can be decomposed into fluctuations in the time reference of the autonomous system, mostly expressed in terms of *phase noise* for analog applications, and perturbations of the amplitude of the generated signal, the *amplitude* or *orbital noise* component. From a practical standpoint, in the vast majority of cases the attention of the designer is focused on the minimization of phase noise only, since this is the dominant fluctuation component close to the fundamental oscillator frequency f_0 (and its harmonics). Nevertheless, orbital fluctuations become important at frequencies far away from frequency f_0 and, in particular in presence of a strong adjacent channel, they might also limit the dynamic range of the receiver [1].

Many approaches have been developed to analyze the effect of circuit noise sources on the oscillator fluctuations performance: the traditional technique is based on a purely linear perturbative approach, where all the circuit variables are expressed as the sum of the noiseless oscillator value and of the noise-induced perturbation. Fluctuations are then assumed of small-amplitude, thus the circuit is linearized around the noiseless working point (thereby transforming it into a linear time-varying system), and the noise sources are linearly propagated towards the circuit output. This technique has been proved to fail for frequencies very close to the harmonics of the oscillating frequency, where a divergence takes place. The issue has been tackled by the nonlinear perturbative analysis proposed in [2], where however phase noise only was considered.

Recently, we extended the analysis in [2] by deriving a consistent statistical characterization of the entire correlation matrix, i.e. by considering not only phase noise but rather including orbital fluctuations and their correlation with phase noise as well [3]. We present here some results of this analysis, showing that orbital noise may significantly impact on the full oscillator noise spectrum.

II. THE MODELING APPROACH

We consider here an autonomous lumped circuit represented by an ordinary differential equation:

$$\frac{\mathrm{d}\mathbf{x}}{\mathrm{d}t} - \mathbf{f}(\mathbf{x}) = \mathbf{0}, \tag{1}$$

where $\mathbf{x}(t)$ is the circuit state vector of size n. Assuming that (1) admits a non-trivial periodic solution (limit cycle) $\mathbf{x}_\mathrm{S}(t)$ of period $T = 1/f_0$, the noisy oscillator is described here by adding to (1) a set of white Gaussian noise sources $\boldsymbol{\xi}(t)$:

$$\frac{\mathrm{d}\mathbf{z}}{\mathrm{d}t} - \mathbf{f}(\mathbf{z}) = \mathbf{B}(\mathbf{z})\boldsymbol{\xi}(t), \tag{2}$$

where the solution-dependent matrix $\mathbf{B}(\mathbf{z})$ takes into account the possible modulation of the noise generators.

The theory in [2] expresses the noisy oscillator solution as the superposition of a time-shifted version of the limit cycle and of an orbital deviation $\mathbf{y}(t)$

$$\mathbf{z}(t) = \mathbf{x}_\mathrm{S}(t + \alpha(t)) + \mathbf{y}(t),$$

where $\alpha(t)$ is a stochastic process responsible for the oscillator phase noise, while $\mathbf{y}(t)$ corresponds to the *orbital fluctuations* (amplitude noise) of the autonomous circuit. The derivation in [2] shows that $\alpha(t)$ satisfies a nonlinear stochastic equation, while the orbital deviation may, at least for stable oscillators, be interpreted as a small-change perturbation of the limit cycle. This analysis, therefore, has been denoted as nonlinear perturbative approach.

The autocorrelation matrix of the noisy oscillator solution is given by:

$$\begin{aligned}\mathbf{R}_{\mathbf{z},\mathbf{z}}(t,\tau) = \mathrm{E}\left\{\mathbf{z}(t)\mathbf{z}^\dagger(t+\tau)\right\} = \mathbf{R}_{\mathbf{x}_\mathrm{S},\mathbf{x}_\mathrm{S}}(t,\tau)\\ + \mathbf{R}_{\mathbf{x}_\mathrm{S},\mathbf{y}}(t,\tau) + \mathbf{R}_{\mathbf{y},\mathbf{x}_\mathrm{S}}(t,\tau) + \mathbf{R}_{\mathbf{y},\mathbf{y}}(t,\tau)\end{aligned}$$

where $\mathrm{E}\left\{\cdot\right\}$ is the ensemble average operator and † denotes the complex conjugate and transpose operation. The first term

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$\mathbf{R}_{\mathbf{x}_S,\mathbf{x}_S}(t,\tau)$ describes phase noise, and has been discussed in detail in [2], where the effect of orbital deviations is neglected altogether.

As discussed in [3], we were able to prove that even including orbital fluctuations the total noisy output $\mathbf{z}(t)$ is, at least asympotically with the observation time, a stationary stochastic process. This means that, in the frequency domain, a stationary spectrum can be defined, which in turn can be decomposed as

$$\mathbf{S}_{\mathbf{z},\mathbf{z}}(\omega) = \mathbf{S}_{\mathbf{x}_S,\mathbf{x}_S}(\omega) + \mathbf{S}_{\text{corr}}(\omega) + \mathbf{S}_{\mathbf{y},\mathbf{y}}(\omega), \qquad (3)$$

where the partial spectra are the Fourier transforms of the asymptotic values of the correlation functions $\mathbf{R}_{\mathbf{x}_S,\mathbf{x}_S}$, $\mathbf{R}_{\mathbf{x}_S,\mathbf{y}} + \mathbf{R}_{\mathbf{y},\mathbf{x}_S}$ and $\mathbf{R}_{\mathbf{y},\mathbf{y}}$, respectively. An important result proved in [3] is that the phase noise contribution $\mathbf{S}_{\mathbf{x}_S,\mathbf{x}_S}(\omega)$ still is given by the same expression derived in [2], and therefore is characterized by the Floquet adjoint eigenvector $\mathbf{v}_1(t)$ associated to the null Floquet exponent $\mu_1 = 0$ always present when the linear system obtained by linearizing (1) around the steady-state solution $\mathbf{x}_S(t)$ is studied. The mathematical background for this analysis is Floquet theory, presented e.g. in [2], [4]. On the other hand, the second and third term in (3) depend on the remaining $n-1$ Floquet exponents and direct and adjoint eigenvectors associated to the oscillator limit cycle. Closed form expressions can be derived, thus making the noise spectrum evaluation completely straightforward once the following two deterministic analyses are carried out:

1) the oscillator working point $\mathbf{x}_S(t)$ is calculated in the time or frequency domain, along with the oscillation frequency;
2) the oscillator equations are linearized around the limit cycle $\mathbf{x}_S(t)$, and the corresponding linear system is analyzed according to Floquet theory estimating all the n Floquet exponents μ_l, and all the associated direct $\mathbf{u}_l(t)$ and adjoint $\mathbf{v}_l(t)$ Floquet eigenvectors ($l = 1, ..., n$). This evaluation can be, for instance, carried out in the frequency domain with the algorithm in [4].

The noise spectrum components are given by

$$\mathbf{S}_{\mathbf{x}_S,\mathbf{x}_S}(\omega) = \sum_h \tilde{\mathbf{X}}_h \tilde{\mathbf{X}}_h^\dagger \frac{h^2\omega_0^2 c}{\Xi_h^2(\omega)} \qquad (4)$$

$$\mathbf{S}_{\text{corr}}(\omega) = \sum_{l=2}^{n} \sum_{h,j} \left\{ \frac{\left(\mathbf{D}_{lhj}^\dagger + \mathbf{D}_{lhj}\right)\left[\frac{1}{2}h^2\omega_0^2 c - \operatorname{Re}\{\mu_l\}\right]}{\Delta_{lhj}^2(\omega)} + \frac{\mathrm{i}\left(\mathbf{D}_{lhj}^\dagger - \mathbf{D}_{lhj}\right)\left[\omega + j\omega_0 + \operatorname{Im}\{\mu_l\}\right]}{\Delta_{lhj}^2(\omega)} - \frac{\left(\mathbf{D}_{lhj}^\dagger + \mathbf{D}_{lhj}\right)\left[\frac{1}{2}h^2\omega_0^2 c\right]}{\Xi_h^2(\omega)} - \frac{\mathrm{i}\left(\mathbf{D}_{lhj}^\dagger - \mathbf{D}_{lhj}\right)\left[\omega + h\omega_0\right]}{\Xi_h^2(\omega)} \right\} \qquad (5)$$

$$\mathbf{S}_{\mathbf{y},\mathbf{y}}(\omega) = \sum_{l=2}^{n} \sum_{h,j} \left\{ \frac{\left(\mathbf{C}_{lhj}^\dagger + \mathbf{C}_{lhj}\right)\left[\frac{1}{2}h^2\omega_0^2 c - \operatorname{Re}\{\mu_l\}\right]}{\Delta_{lhj}^2(\omega)} + \frac{\mathrm{i}\left(\mathbf{C}_{lhj}^\dagger - \mathbf{C}_{lhj}\right)\left(\omega + j\omega_0 + \operatorname{Im}\{\mu_l\}\right)}{\Delta_{lhj}^2(\omega)} \right\}, \qquad (6)$$

where i is the imaginary unit, $\omega_0 = 2\pi/T$ is the angular frequency of oscillation and $\tilde{\mathbf{X}}_h$ is th h-th harmonic amplitude of the (exponential) Fourier representation of $\mathbf{x}_S(t)$. The other coefficients are (here $\mathbf{B}^{\mathrm{T}}$ denotes the transpose of $\mathbf{B}$)

$$c = \frac{1}{T}\int_0^T \mathbf{v}_1^{\mathrm{T}} \mathbf{B}\mathbf{B}^{\mathrm{T}} \mathbf{v}_1 \,\mathrm{d}t \qquad (7)$$

$$\Xi_h^2(\omega) = \left[\frac{1}{2}h^2\omega_0^2 c\right]^2 + \left[\omega + h\omega_0\right]^2 \qquad (8)$$

$$\Delta_{lhj}^2(\omega) = \left[\frac{1}{2}h^2\omega_0^2 c - \operatorname{Re}\{\mu_l\}\right]^2 + \left[\omega + j\omega_0 + \operatorname{Im}\{\mu_l\}\right]^2 \qquad (9)$$

$$\mathbf{C}_{lhj} = \sum_{l'=2}^{n} \sum_{j'} \frac{\tilde{\mathbf{U}}_{l'_{j'}} \tilde{\mathbf{\Lambda}}_{l'_{h-j'}}^{\mathrm{T}} \tilde{\mathbf{\Lambda}}_{l_{h-j}}^{*} \tilde{\mathbf{U}}_{l_j}^\dagger}{\mathrm{i}(j-j')\omega_0 - \mu_{l'} - \mu_l^*} \qquad (10)$$

$$\mathbf{D}_{lhj} = \tilde{\mathbf{X}}_h \tilde{\mathbf{V}}_{1_0}^{\mathrm{T}} \tilde{\mathbf{\Lambda}}_{l_{h-j}}^{*} \tilde{\mathbf{U}}_{l_j}^\dagger \frac{\mathrm{i}h\omega_0}{-\mu_l^* - \mathrm{i}(h-j)\omega_0}. \qquad (11)$$

In the previous expressions $\tilde{\mathbf{V}}_{1_0}$ is the DC harmonic component of $\mathbf{v}_1(t)^{\mathrm{T}}\mathbf{B}(t)$, and $\tilde{\mathbf{U}}_{l_j}$ and $\tilde{\mathbf{\Lambda}}_{l_k}^{\mathrm{T}}$ are the Fourier coefficients of $\mathbf{u}_l(t)$ and $\mathbf{v}_l^{\mathrm{T}}(t)\mathbf{B}(t)$, respectively.

Since the presence of a complex Floquet exponent $\mu_{l'}$ requires that the complex conjugate $\mu_{l'}^* = \mu_{l''}$ is also a Floquet exponent, the summations in the correlation and orbital spectra yield resonance effects in the total oscillator noise spectrum as discussed in [1]. Furthermore, $\mathbf{C}_{lhj}$ and $\mathbf{D}_{lhj}$ are related to $1/\mu_l$, thus suggesting that the contribution of the orbital deviations might be more significant for oscillators whose limit cycle is characterized by at least a second Floquet exponent near to zero, such as the class of high-Q oscillators [5].

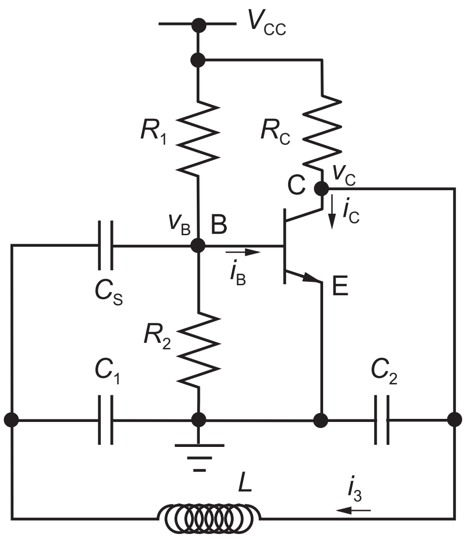

Fig. 1: Circuit of the Colpitts oscillator.

III. Results

We discuss first the Colpitts oscillator presented in Fig. 3, where the bipolar transistor is represented by a memoryless, Gummel-Poon simplified model with a forward DC bias of 100. The circuit parameters are: $V_{\mathrm{CC}} = 15$ V, $R_1 = 400$ kΩ, $R_2 = 71.429$ kΩ, $R_{\mathrm{C}} = 4.9$ kΩ, $C_1 = 300$ nF, $C_2 = 9.09$ nF, $C_{\mathrm{S}} = 1$ μF and $L = 27.78$ nH. The circuit has been analyzed with the harmonic balance technique including 60 harmonics, obtaining $f_0 = 10$ MHz.

The noise calculations assume for simplicity that only the transistor is noisy, and affected by white shot noise. The collector current noise spectrum, in $\mathrm{dB_W}$/Hz, is reported in Fig. 2 as a function of the upper (i.e., $f > f_0$) and lower (i.e., $f < f_0$) sidebands of the fundamental frequency. Notice the effect of the complex Floquet exponents, which yield the small resonance-like peak 11.7 kHz away from the fundamental.

The second example is the common-base, negative-resistance oscillator shown in Fig. 3. The scheme, proposed in [6], is based on a InGaP/GaAs HBT represented by the Gummel-Poon model described in [6]. The circuit connected to the emitter has been designed to allow oscillations to build up at 5 GHz, and was simulated in the frequency domain including 30 harmonics plus DC. The calculated total noise spectrum, and its components according to (3), are shown in Fig. 4 as a function of the absolute frequency. Again the correlation component is negligible, while orbital noise becomes the dominant term for frequencies away from the harmonics. The circuit admits 6 Floquet exponent, whose values are listed in Table I.

The partial contributions to the orbital noise spectrum due to the 5 non null Floquet exponents are plotted in Fig. 5. As expected, the dominant contribution is that of the exponent with smallest real part. Notice however that

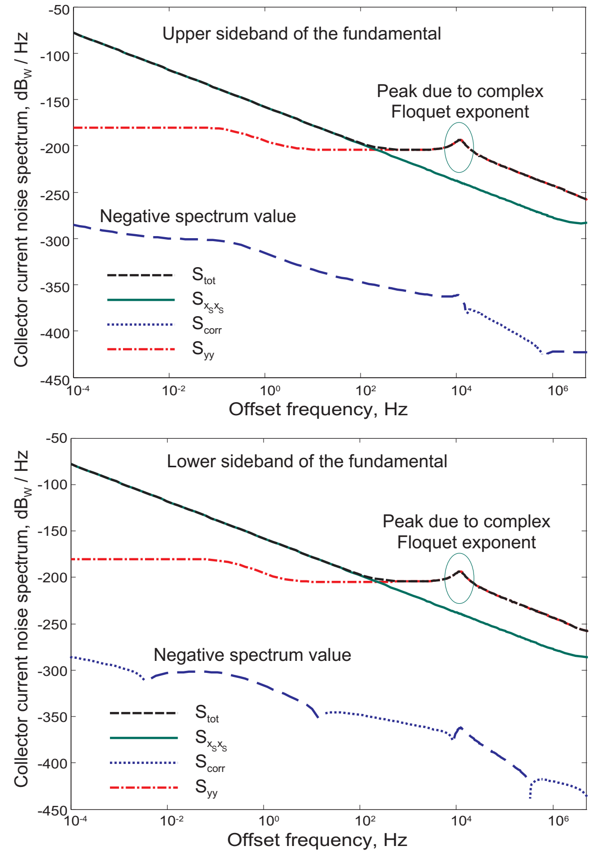

Fig. 2: Upper (above) and lower (below) sideband frequency dependence of the collector current noise spectrum of the Colpitts oscillator around the fundamental frequency f_0.

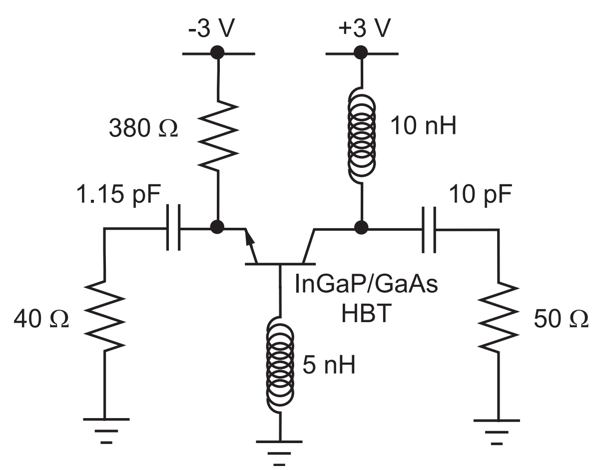

Fig. 3: Circuit of the InGaP/GaAs oscillator in [6].

the second largest component is due to μ_5, thus pointing out that the contribution of the Floquet eigenvectors might be more important than the pure $1/\mu_l$ term.

The sideband representation of the noise spectrum around the fundamental and the second harmonic component are shown in Fig. 6 and 7, respectively. Even in this case, the orbital contributions are responsible of asymmetries in the spectrum around the oscillator harmonics.

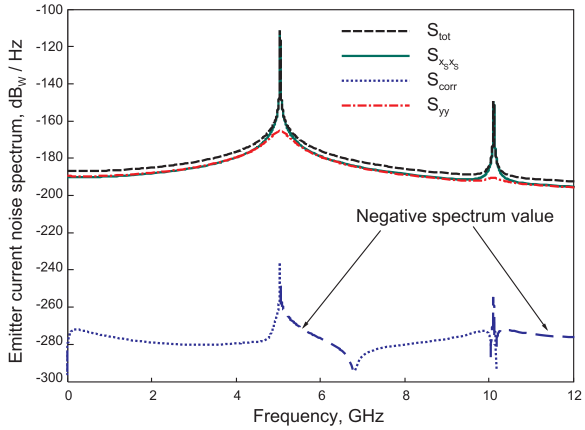

Fig. 4: Total noise spectrum of the InGaP/GaAs oscillator as a function of the absolute frequency.

Table I: Floquet's exponents for the HBT oscillator

Exponent	Value [s^{-1}]
μ_1	0
μ_2	-1.27×10^9
μ_3	$-2.50 \times 10^9 + i1.94 \times 10^9$
μ_4	$-2.50 \times 10^9 - i1.94 \times 10^9$
μ_5	-1.14×10^{10}
μ_6	-4.71×10^{11}

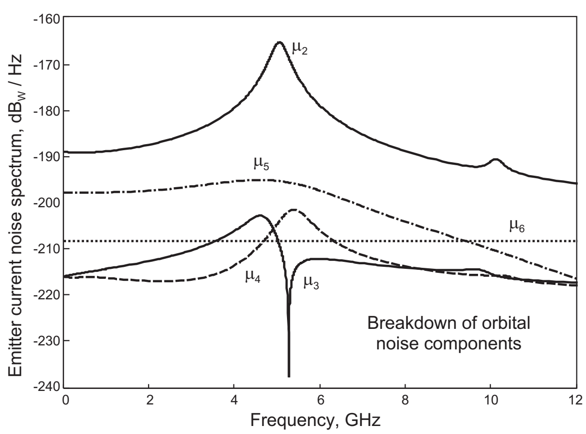

Fig. 5: Partial contributions to the orbital noise spectrum of the InGaP/GaAs oscillator.

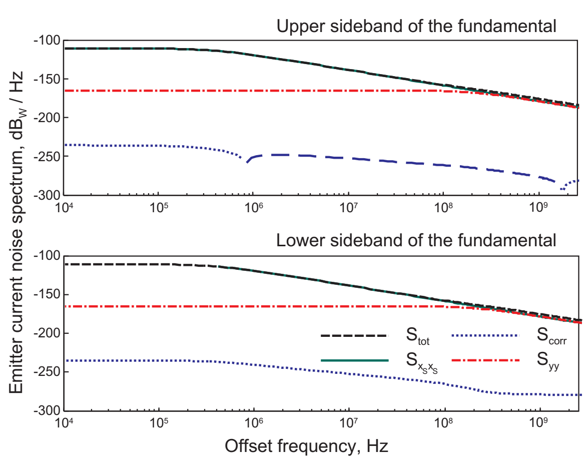

Fig. 6: Upper and lower sideband total noise spectrum of the InGaP/GaAs oscillator around the fundamental.

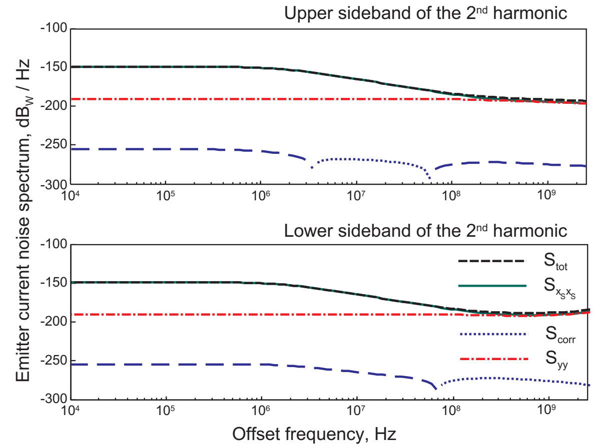

Fig. 7: Upper and lower sideband total noise spectrum of the InGaP/GaAs oscillator around the second harmonic.

IV. Conclusions

We have presented closed-form expression for the total noise spectrum of a free running oscillator where (modulated) white noise sources are present. The model is based on a rigorous analysis which extends the results in [2], where phase noise only was considered. The expressions can be easily implemented into available EDA tools provided that Floquet analysis is made available.

The modeling approach has been applied to two examples of oscillators, showing that the orbital contribution can provide significant effects to the noise spectrum shape, at least sufficiently far away from the solution harmonics.

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